SYMBOLIC LOGIC

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Second Edition

[1932] 1959

DOVER PUBLICATIONS, INC.

APPENDIX I

THE USE OF DOTS AS BRACKETS

There are a number of points with regard to the punctuation or bracketing of logical expressions which could not be conveniently explained at any one place in the text, because the symbols of different sorts, in connection with which the use of bracketing dots requires explanation, were not all introduced at once. So, we propose to deal here somewhat in detail with the technique of the use of dots as brackets and to explain in order the several points involved. The scheme that we have adopted is that of *Principia Mathematica*, which, as will be seen, includes some conventions that are essential to any system of bracketing and others that are arbitrary.

Consider, in the first place, an expression like

$$(p \supset p') \vee (q \supset q'),$$

where the use of brackets, or some equivalent scheme of punctuation, is essential. It will be seen, in the case of this particular expression, that we can if we like dispense with the two extreme brackets without incurring ambiguity; we can write

$$p \supset p') \vee (q \supset q'.$$

And then we can replace the remaining brackets by dots:

$$p \supset p' . \vee . q \supset q'.$$

But if this is to be done, one point must be carefully noted: the brackets are asymmetrical, and thus indicate the direction in which they operate, whereas the dots are ambiguous in this respect; so that we must introduce the convention that a dot is always to be understood as operating away from the connective or other symbol beside which it occurs.

Suppose, now, that we have such an expression as

$$[(p \vee p') \supset (q \vee q')] \vee [(r \vee r') \supset (s \vee s')],$$

which involves brackets of two sorts, wider and narrower. It is

easy to see that we can write, without ambiguity,

$$p \vee p') \supset (q \vee q')] \vee [(r \vee r') \supset (s \vee s';$$

and if we adopt the convention that a square bracket is to be understood as being stronger than a round one, in the sense that a square bracket can have within its scope a round one, but not conversely, then we can write:

$$p \vee p') \supset (q \vee q'] \vee [r \vee r') \supset (s \vee s'.$$

We can then replace square brackets by double dots and round brackets by single ones:

$$p \vee p' . \supset . q \vee q' : \vee : r \vee r' . \supset . s \vee s'.$$

The scope of a single dot is to be understood as being closed by another single dot, or by two dots, whereas the scope of two dots is not closed by a single one. Thus, in place of

$$(p \supset p') \vee (q \supset q') \vee (r \supset r'),$$

we can write

$$p \supset p' . \vee . q \supset q' . \vee . r \supset r',$$

or

$$p . \supset . p' : \vee : q . \supset . q' : \vee : r . \supset . r'.$$

And these same conventions are to be extended to the use of any number of dots that may be required.

Now, let us take the two expressions:

$$(p) \supset (q \vee r) \quad \text{and} \quad p \supset (q \vee r),$$

where, of course, brackets around the p in the first expression are not strictly necessary; and let us replace brackets by dots, so that we have

$$p . \supset . q \vee r \quad \text{and} \quad p \supset . q \vee r.$$

Here, although one of the dots in the first expression is superfluous, we commonly put it in for the sake of symmetry. Again, let us take the three expressions:

$$p \supset [(q \vee r) \supset (q \vee r)],$$
$$[p] \supset [(q \vee r) \supset (q \vee r)],$$
$$(p) \supset [(q \vee r) \supset (q \vee r)],$$

in connection with which, when brackets are replaced by dots, we have

$$p \supset : q \vee r . \supset . q \vee r,$$

$$p : \supset : q \vee r . \supset . q \vee r,$$

$$p . \supset : q \vee r . \supset . q \vee r.$$

In accordance with our conventions, each of these expressions is correct, the first being the most economical in the use of dots. The second has the advantage that coördinate elements in the complex are punctuated alike; but this is really unnecessary, and it is common practice to adopt the third form. Whenever one or more dots are required on one side of a connective, we put one dot on the other side even when it is not necessary to do so.

We have, so far, omitted consideration of expressions involving conjunction, because we also use dots to mean 'and,' so that the conventions in this respect require special discussion. Let us take the expression

$$(p \vee q) . (r \vee s),$$

in which the dot stands only for conjunction. If we here replace brackets by dots, as above, we have

$$p \vee q \; . \; . \; . \; r \vee s,$$

where the middle dot is the sign of conjunction and the outer ones serve as brackets. But it is customary, in such a case, simply to write

$$p \vee q . r \vee s,$$

and thus to allow a single dot to stand for conjunction and do the work of bracketing as well. When this is done, however, it is to be noted that the bracketing dot used, unlike the others we have considered, must be understood as operating in both directions, since it does the work of two dots, one on each side of the sign of conjunction.

Again, consider such an expression as

$$[(p \supset q) . (q \supset p)] \vee [(r \supset s) . (s \supset r)],$$

which, when brackets are replaced by dots in accordance with the conventions so far put down, becomes

$$p \supset q . q \supset p : \vee : r \supset s . s \supset r.$$

Now, it is not customary in such a case to use double dots in connection with $\vee$; we may write

$$p \supset q . q \supset p . \vee . r \supset s . s \supset r$$

and understand the dots which stand also for conjunction as being 'weaker' than those occurring beside a connective—as if some of their force were taken up in meaning 'and.' In place of $(p . q) \vee r$ we write $p . q . \vee . r$ rather than $p . q : \vee . r$; and in place of $p . [q \supset (r \vee s)]$ we write $p : q . \supset . r \vee s$. In general, a given number of dots occurring beside a connective will carry over an equal number standing also for conjunction, and a given number standing for conjunction will carry over a lesser number beside a connective. And, of course, a single dot meaning conjunction will carry over a bare connective—as in $p . q \vee r$, which means $p . (q \vee r)$.

There is one further matter that must be dealt with regarding the way dots are to be understood as operating in given directions. Consider the expression

$$s \vee [p \vee \{(q \supset r) \supset s\}],$$

which we should write

$$s . \vee :. p . \vee : q \supset r . \supset . s,$$

and note that we here use three dots after the first occurrence of $\vee$ in order to get over the two dots which come after the second occurrence, despite the fact that these latter dots operate in the same direction as the set of three. That is to say, we regard the scope of a set of dots as being closed when another set of equal or greater strength is met, no matter what the direction of operation of the other set. It would be possible to adopt a convention to the effect that the scope of a set of dots could be closed only by dots operating in the opposite direction; and in accordance with such a convention, we could write

$$s . \vee : p . \vee : q \supset r . \supset . s,$$

which would correspond to

$$s \vee \{p \vee \{(q \supset r) \supset s\}\}.$$

But this would make our expressions less easy to read and would effect a very slight compensating economy.

There remains only one other class of dots to be discussed, namely, those which come after prefixes. Consider, for example, the expression

$$(x)[f(x) . f'(x)],$$

which is to be written

$$(x) . f(x) . f'(x).$$

We regard a dot which comes after a prefix as being stronger than one which indicates conjunction. But, on the other hand, we take such a dot to be weaker than one standing beside a connective; so that, for example,

$$p \vee [(x) . f(x)]$$

may be written

$$p . \vee . (x) . f(x),$$

where the single dot after $\vee$ brackets the entire expression to the right of it. We write

$$(x) : (\exists y) . f(x) . f'(y)$$

in place of

$$(x)[(\exists y)\{f(x) . f'(y)\}],$$

and

$$(x) . f(x) : (\exists y) . f'(y)$$

in place of

$$[(x)\{f(x)\}] . [(\exists y)\{f'(y)\}].$$

Now, inasmuch as $\sim$ and $\Diamond$ are prefixes, it might be expected that we should deal with them as we do with (x) and $(\exists x)$; that we should, for example, replace

$$\sim[(\exists x) . f(x)] \quad \text{and} \quad \Diamond[(\exists x) . f(x)]$$

by

$$\sim:(\exists x) . f(x) \quad \text{and} \quad \Diamond:(\exists x) . f(x),$$

and replace

$$\sim(p \vee q . \supset . p) \quad \text{and} \quad \Diamond(p \vee q . \supset . p)$$

by

$$\sim:p \vee q . \supset . p \quad \text{and} \quad \Diamond:p \vee q . \supset . p.$$

This procedure would be quite in accordance with the conventions we have adopted; but, in fact, we dispose of these two cases

in other ways. In the first case, we write

$$\sim(\exists x) . f(x) \quad \text{and} \quad \Diamond(\exists x) . f(x),$$

without either brackets or dots—as if $\sim(\exists x)$ and $\Diamond(\exists x)$ were single operators, although of course they are not, since the scope of the attached prefix is, in each case, the entire expression $(\exists x) . f(x)$. In the same way, we write

$$(x) : \sim(\exists y) . f(x, y) \quad \text{and} \quad (x) : \Diamond(\exists y) . f(x, y)$$

in place of

$$(x) . \sim[(\exists y) . f(x, y)] \quad \text{and} \quad (x) . \Diamond[(\exists y) . f(x, y)];$$

and, again,

$$\sim\Diamond(\exists x) . f(x)$$

in place of

$$\sim[\Diamond\{(\exists x) . f(x)\}].$$

In the second case, where we have $\sim(p \vee q . \supset . p)$ and $\Diamond(p \vee q . \supset . p)$, it is customary simply to retain the brackets, rather than to make use of dots.

It will be observed that nearly all the foregoing examples illustrate the minimum number of dots that may be used in a given case. We must therefore point out that the scheme is more elastic than would appear from these illustrations. It is always possible, in the interest of clarity or of emphasis, to use more dots than are strictly required, provided, of course, that we have due regard for the different degrees of strength of dots of different classes. Thus, $(x) . f(x) . g(x)$ and $(x) : f(x) . g(x)$ mean the same thing, as do $p . \supset . (\exists x) . f(x)$ and $p . \supset : (\exists x) . f(x)$.